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BEAMS ON A TWO-DIMENSIONAL PASTERNAK BASE SUBJECTED TO LOADS THAT CAUSE LIFT-OFF

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Abstract—First, two closely related problems, shown in Fig. 1, published by Chernigovskaya (1961, In *Issledovaniya po Dinamike Sooruzhenii i Raschetu Konstruktsii na Uprugom Osnovanii* (Edited by B. G. Korenev, pp. 113–141. Gosstroiizdat, Moscow) and Ting (1973, *J. Franklin Inst.* **296**(2), 77–89) are discussed. It is shown that neither of their formulations is correct. The aim of this paper is to show how to correctly formulate and solve problems of this type. Utilizing the variational approach for variable matching points derived by Kerr (1976, *Int. J. Solids Structures* **12**, 1–11), a formulation for the problem analyzed by Ting is presented, that is mechanically reasonable and mathematically well posed. The analytical solution obtained is evaluated numerically and then compared with related test results by Durelli *et al.* (1969, *J. Struct. Div. ASCE* **95**, 1713–1725). This paper concludes with a discussion of the results obtained.

INTRODUCTION

The analytical aspects of unbonded contact problems for continuously supported structures have been discussed by Kerr (1976, 1979), when the base response is represented by a Pasternak foundation model. For problems of this type the contact region is not known *a priori*. Consequently, the matching conditions at the point of separation must include an additional equation for the determination of this unknown.

As an example, Kerr (1976) considered a finite beam resting on a two-dimensional Pasternak foundation and subjected to vertical loads such that lift-off of the beam is possible, as shown in Fig. 1. Developing variational calculus for variable matching points, he showed that this problem was incorrectly formulated by Chernigovskaya (1961). Solving the case when $q(x) = \text{const}$ and $P = 0$, she heuristically prescribed the following matching conditions at the separation point, $x = l$:

$$\left.\begin{aligned} -EIw''(l) &= -q(L-l)^2/2 \\ -EIw'''(l) &= q(L-l) \\ kw(l)-Gw''(l) &= 0 \end{aligned}\right\} \tag{1}$$

where $w(x)$ is the vertical deflection, $w' = dw/dx$, G is the parameter of the shear layer, k is the parameter of the spring layer, q is the uniform load, $2l$ is the length of contact of base and beam and $2L$ is the length of the beam. For the case of a plate strip in cylindrical bending with $w = w(x)$, EI is replaced by $D = Eh^3/[12(1-\nu^2)]$.

For the two-dimensional Pasternak foundation model the contact pressure is (Kerr, 1964)

$$p(x) = kw(x) - Gw''(x). \tag{2}$$

Thus, the third condition in (1) prescribes that the pressure is zero at the point of separation. This is not correct. The proper condition, which results from the variational formulation, is that at $x = l$ the slope of the shear layer is continuous. Therefore, the contact pressure at the separation point may have a non-zero value. Kerr (1979) pointed out that because of the reduced order of the differential equation that governs the Pasternak foundation response (as compared to the elastic continuum) only one of the two anticipated conditions

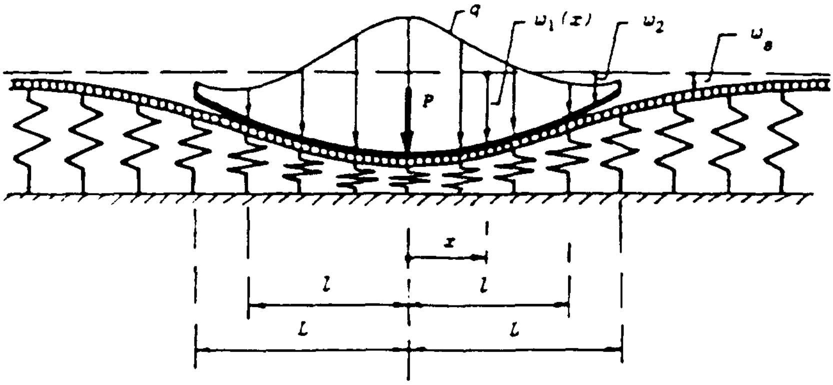

Fig. 1. Beam problem under consideration.

may be satisfied. For the problem under consideration the tangency condition is the one to be retained, and not the zero-pressure condition listed in (1).

A similar problem was also incorrectly formulated later by Ting (1973). He solved the problem shown in Fig. 1 for the case of $q(x) = 0$, by prescribing the following conditions at $x = l$:

$$\left.\begin{aligned} w(l) &= 0 \\ w''(l) &= 0 \\ w'''(l) &= 0 \end{aligned}\right\} . \tag{3}$$

This formulation incorrectly stipulates that the deflection at the separation point is zero. Whereas this is true for the Winkler foundation, it does not apply to the Pasternak base model. As a consequence of the first two conditions in (3), Ting's formulation implies a zero contact pressure at the separation point $x = l$. This is not correct for the reason given in discussing the Chernigovskaya formulation. Another major error in Ting's formulation is his omission of the differential equation for the base layer beyond the point of separation.

In the following the correct formulation of Ting's problem is stated and then solved. The closed-form solution obtained is then numerically evaluated and the results are compared with those of the photoelastic results reported by Durelli *et al.* (1969).

FORMULATION OF PROBLEM

The problem to be analyzed is shown in Fig. 1 with $q(x) = 0$. It consists of an elastic beam resting on a two-dimensional Pasternak foundation that is subjected at the center to a vertical concentrated load, P. According to the derivation by Kerr (1976), utilizing symmetry, assuming the validity of beam bending theory, and denoting $w_1(x)$ as the vertical deflection of the beam axis in $0 < x < l$, $w_2(x)$ as the vertical deflection of the beam axis in $l < x < L$, and $w_s(x)$ as the vertical deflection of the shear layer in $l < x < \infty$, for $EI = \text{const}$ and $G = \text{const}$, the following differential equations result:

$$\left.\begin{aligned} & EIw_1^{\text{IV}} - Gw_1'' + kw_1 = 0 \quad 0 < x < l \\ & EIw_2^{\text{IV}} = 0 \quad l < x < L \\ & Gw_s'' - kw_s = 0 \quad l < x < \infty \end{aligned}\right\} . \tag{4}$$

The corresponding boundary and matching conditions are:

$$\left.\begin{aligned} w_1'(0) &= 0 \\ w_1'''(0) &= P/(2EI) \end{aligned}\right\} \tag{5}$$

$$\left.\begin{aligned} w_1(l) &= w_2(l) \\ w_1(l) &= w_3(l) \\ w_1'(l) &= w_2'(l) \\ w_1'(l) &= w_3'(l) \\ w_1''(l) &= w_2''(l) \\ w_1'''(l) &= w_2'''(l) \end{aligned}\right\} \tag{6}$$

$$\left.\begin{aligned} w_2''(L) &= 0 \\ w_2'''(L) &= 0 \end{aligned}\right\} \tag{7}$$

$$\lim_{x \to \infty} \{w_3\} \to \text{finite}. \tag{8}$$

These are the 11 boundary conditions for the determination of the 10 integration constants and the, as yet, unknown length, l.

This completes the formulation of the problem under consideration. It is very different to the one presented by Ting (1973). Next, the above formulation is solved.

SOLUTION OF FORMULATION

The differential equations contain only constant coefficients. Therefore, the general solutions are of the form $w(x) = A\,e^{mx}$. Substituting this into the first differential equations gives

$$m^4 - \frac{G}{EI}m^2 + \frac{k}{EI} = 0. \tag{9}$$

The roots of this equation are

$$m_{1,2,3,4} = \pm\sqrt{\frac{G}{2EI} \pm \sqrt{\left(\frac{G}{2EI}\right)^2 - \frac{k}{EI}}}. \tag{9'}$$

Three cases may be distinguished; namely $G \lesseqgtr 2\sqrt{kEI}$.

At first, the case $G < 2\sqrt{kEI}$ is analyzed. The general solutions of the three differential equations in (4) are

$$\left.\begin{aligned} w_1(x) &= [A_1 \cos(\rho x) + A_2 \sin(\rho x)]\cosh(\kappa x) + [A_3 \cos(\rho x) + A_4 \sin(\rho x)]\sinh(\kappa x) \\ w_2(x) &= A_5 x^3 + A_6 x^2 + A_7 x + A_8 \\ w_3(x) &= A_9\,e^{-\mu x} + A_{10}\,e^{\mu x}, \end{aligned}\right\} \tag{10}$$

where

$$\left.\begin{matrix}\kappa \\ \rho\end{matrix}\right\} = \sqrt{\sqrt{\frac{k}{4EI}} \pm \frac{G}{4EI}}; \quad \mu = \sqrt{\frac{k}{G}}. \tag{11}$$

The 10 integration constants $A_1, \ldots, A_{10}$ and the length l are determined by substituting the expressions for the ws into the boundary conditions (5)–(8). They are :

$$
\begin{aligned}
A_1 &= \frac{P}{4EI\kappa\rho(\rho^2+\kappa^2)^2\psi}\{-\kappa^2(\rho^2+\kappa^2)\sin^2(\rho l)+\rho^2(\rho^2+\kappa^2)\sinh^2(\kappa l)+4\rho^2\kappa^2\} \\
A_2 &= \frac{P}{4EI\rho(\rho^2+\kappa^2)}; \quad A_3 = \frac{-P}{4EI\kappa(\rho^2+\kappa^2)} \\
A_4 &= \frac{P}{4EI(\rho^2+\kappa^2)^2\psi}\{(\rho^2+\kappa^2)[\cos^2(\rho l)-\cosh^2(\kappa l)]-2(\kappa^2-\rho^2)\} \\
A_5 &= 0; \quad A_6 = 0 \\
A_7 &= \frac{P}{2EI(\rho^2+\kappa^2)\psi}\{\rho\cos(\rho l)\sinh(\kappa l)-\kappa\sin(\rho l)\cosh(\kappa l)\} \\
A_8 &= \frac{P}{2EI(\rho^2+\kappa^2)^2\psi}\{2\kappa\rho\cos(\rho l)\cosh(\kappa l)-(\kappa^2-\rho^2)\sin(\rho l)\sinh(\kappa l) \\
&\qquad +(\rho^2+\kappa^2)[\kappa l\sin(\rho l)\cosh(\kappa l)-\rho l\cos(\rho l)\sinh(\kappa l)]\} \\
A_9 &= \frac{P\,e^{\mu l}}{2EI(\rho^2+\kappa^2)^2\psi}\{2\kappa\rho\cos(\rho l)\cosh(\kappa l)-(\kappa^2-\rho^2)\sin(\rho l)\sinh(\kappa l)\} \\
A_{10} &= 0,
\end{aligned}
\tag{12}
$$

where

$$\psi = \kappa\cos(\rho l)\sin(\rho l)+\rho\cosh(\kappa l)\sinh(\kappa l). \tag{13}$$

The condition for the determination of l is

$$
\begin{aligned}
&\mu[(\kappa^2-\rho^2)\sin(\rho l)\sinh(\kappa l)-2\kappa\rho\cos(\rho l)\cosh(\kappa l)] \\
&\qquad +(\kappa^2+\rho^2)[\kappa\sin(\rho l)\cosh(\kappa l)-\rho\cos(\rho l)\sinh(\kappa l)] = 0.
\end{aligned}
\tag{14}
$$

The corresponding expressions for the bending moments, beam shearing forces and the pressure distribution in the contact region are:

$$
\begin{aligned}
M(x) &= -EIw_1''(x) \\
&= \frac{P}{4\rho\kappa}\left\{\frac{1}{\psi}\{\rho\kappa[\cos^2(\rho l)+\cosh^2(\kappa l)]\sin(\rho x)\sinh(\kappa x)+[\kappa^2\sin^2(\rho l)\right. \\
&\left.+\rho^2\sinh^2(\kappa l)]\cos(\rho x)\cosh(\kappa x)\}-\kappa\sin(\rho x)\cosh(\kappa x)-\rho\cos(\rho x)\sinh(\kappa x)\right\}
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
V(x) &= -EIw_1'''(x) \\
&= \frac{P}{4\kappa\rho}\left\{\frac{1}{\psi}\{\rho[2\kappa^2\cos^2(\rho l)+(\kappa^2-\rho^2)\sinh^2(\kappa l)]\sin(\rho x)\cosh(\kappa x)\right. \\
&\qquad +\kappa[2\rho^2\cosh^2(\kappa l)+(\kappa^2-\rho^2)\sin^2(\rho l)]\cos(\rho x)\sinh(\kappa x)\} \\
&\qquad \left.-(\kappa^2-\rho^2)\sin(\rho x)\sinh(\kappa x)-2\kappa\rho\cos(\rho x)\cosh(\kappa x)\right\}
\end{aligned}
\tag{16}
$$

and

$$p(x) = kw_1(x) - Gw_1''(x)$$
$$= \frac{PG}{4\sqrt{kEI}}\left\{\frac{1}{\psi}\left(\left[z_1 \sinh^2(\kappa l) - z_2 \sin^2(\rho l) + 2\left(\frac{\rho^2 z_2 + \kappa^2 z_1}{\kappa^2+\rho^2}\right)\right]\sin(\rho x)\sinh(\kappa x)\right.\right.$$
$$\left. + \left[\frac{\kappa}{\rho} z_1 \sin^2(\rho l) + \frac{\rho}{\kappa} z_2 \sinh^2(\kappa l) + \frac{4\kappa\rho\mu^2}{\kappa^2+\rho^2}\right]\cos(\rho x)\cosh(\kappa x)\right)$$
$$\left. - \frac{z_1}{\rho}\sin(\rho x)\cosh(\kappa x) - \frac{z_2}{\kappa}\cos(\rho x)\sinh(\kappa x)\right\}. \quad (17)$$

where

$$z_1 = \rho^2 + \kappa^2 - \mu^2, \quad z_2 = \rho^2 + \kappa^2 + \mu^2. \quad (18)$$

Next, the case $G > 2\sqrt{kEI}$ is analyzed. The solutions of the three differential equations in (4) are

$$\begin{aligned} w_1(x) &= B_1 \sinh(\alpha x) + B_2 \cosh(\alpha x) + B_3 \sinh(\beta x) + B_4 \cosh(\beta x) \\ w_2(x) &= B_5 x^3 + B_6 x^2 + B_7 x + B_8 \\ w_3(x) &= B_9 e^{-\mu x} + B_{10} e^{\mu x} \end{aligned} \quad (19)$$

where

$$\left.\begin{matrix}\alpha \\ \beta\end{matrix}\right\} = \sqrt{\frac{G}{2EI} \pm \sqrt{\left(\frac{G}{2EI}\right)^2 - \frac{k}{EI}}}; \quad \mu = \sqrt{\frac{k}{G}}. \quad (20)$$

The integration constants $B_1, \ldots, B_{10}$ determined are

$$\begin{aligned}
B_1 &= \frac{P}{2EI\alpha(\alpha^2-\beta^2)} \\
B_2 &= \frac{P}{2EI\alpha^2(\alpha^2-\beta^2)\phi}[\alpha^2\cosh(\alpha l)\cosh(\beta l) - \alpha\beta\sinh(\alpha l)\sinh(\beta l) - \beta^2] \\
B_3 &= \frac{-P}{2EI\beta(\alpha^2-\beta^2)} \\
B_4 &= \frac{P}{2EI\beta^2(\alpha^2-\beta^2)\phi}[\beta^2\cosh(\alpha l)\cosh(\beta l) - \alpha\beta\sinh(\alpha l)\sinh(\beta l) - \alpha^2] \\
B_5 &= 0; \quad B_6 = 0 \\
B_7 &= \frac{P}{2EI\beta\alpha\phi}[\beta\sinh(\alpha l) - \alpha\sinh(\beta l)] \\
B_8 &= \frac{P}{2EI\beta^2\alpha^2\phi}\{\beta^2\cosh(\alpha l) - \alpha^2\cosh(\beta l) + \alpha\beta l[\alpha\sinh(\beta l) - \beta\sinh(\alpha l)]\} \\
B_9 &= \frac{P\, e^{\mu l}}{2EI\beta^2\alpha^2\phi}[\beta^2\cosh(\alpha l) - \alpha^2\cosh(\beta l)] \\
B_{10} &= 0,
\end{aligned} \quad (21)$$

where

$$\phi = \beta \cosh(\alpha l)\sinh(\beta l) - \alpha \cosh(\beta l)\sinh(\alpha l) \tag{22}$$

and the condition for the determination of l is

$$\alpha\beta[\alpha \sinh(\beta l) - \beta \sinh(\alpha l)] + \mu[\alpha^2 \cosh(\beta l) - \beta^2 \cosh(\alpha l)] = 0. \tag{23}$$

The corresponding bending moments, shearing forces in the beam and contact pressure distribution are

$$M(x) = \frac{-P}{2(\alpha^2-\beta^2)}\left\{\alpha \sinh(\alpha x) - \beta \sinh(\beta x) + \frac{1}{\phi}\{[\alpha^2 \cosh(\alpha l)\cosh(\beta l) - \alpha\beta \sinh(\alpha l)\sinh(\beta l) - \beta^2]\cosh(\alpha x) + [\beta^2 \cosh(\alpha l)\cosh(\beta l) - \alpha\beta \sinh(\alpha l)\sinh(\beta l) - \alpha^2]\cosh(\beta x)\}\right\} \tag{24}$$

$$V(x) = \frac{-P}{2(\alpha^2-\beta^2)}\left\{\alpha^2 \cosh(\alpha x) - \beta^2 \cosh(\beta x) + \frac{1}{\phi}\{\alpha[\alpha^2 \cosh(\alpha l)\cosh(\beta l) - \alpha\beta \sinh(\alpha l)\sinh(\beta l) - \beta^2]\sinh(\alpha x) + \beta[\beta^2 \cosh(\alpha l)\cosh(\beta l) - \alpha\beta \sinh(\alpha l)\sinh(\beta l) - \alpha^2]\sinh(\beta x)\}\right\} \tag{25}$$

$$p(x) = \frac{PG}{2EI(\alpha^2-\beta^2)}\left\{\alpha[(\mu/\alpha)^2 - 1]\sinh(\alpha x) - \beta[(\mu/\beta)^2 - 1]\sinh(\beta x) + \frac{1}{\phi}\{[(\mu/\alpha)^2 - 1][\alpha^2 \cosh(\alpha l)\cosh(\beta l) - \alpha\beta \sinh(\alpha l)\sinh(\beta l) - \beta^2]\cosh(\alpha x) + [(\mu/\beta)^2 - 1][\beta^2 \cosh(\alpha l)\cosh(\beta l) - \alpha\beta \sinh(\alpha l)\sinh(\beta l) - \alpha^2]\cosh(\beta x)\}\right\}. \tag{26}$$

This completes the solution to the problem under consideration. The case $G = 2\sqrt{kEI}$ is not of interest for this study. Next, the solution obtained is compared with experimental results.

COMPARATIVE STUDY

Available experimental results

The only results located related to the problem under consideration are those of photoelastic tests conducted by Durelli *et al.* (1969), which were used by Ting (1973). Durelli *et al.* tested beams of different lengths resting on an elastic foundation and subjected

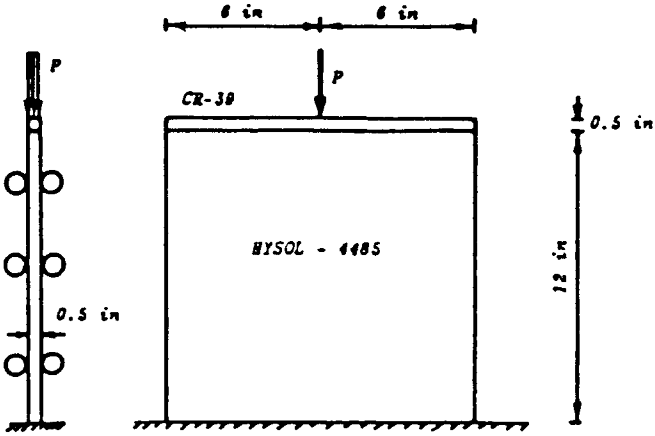

Fig. 2. Test set-up by Durelli *et al.* (1969) (1 in = 2.54 cm).

to various loadings. Their experimental set-up is shown in Fig. 2. It consisted of a beam made of a hard transparent plastic (CR-39) resting on a slab of soft transparent polyurethane rubber (HYSOL 4485) of the same width as the beam. The beam was subjected to a vertical load, $P = 140$ N (31.46 lb), placed in the center.

The elastic properties for these materials are

$$\left.\begin{array}{lll} \text{CR-39} & E = 2.22 \times 10^6\ \text{N cm}^{-2}\ (3.22 \times 10^6\ \text{psi}) \\ & \nu = 0.42 \\ \text{HYSOL-4485} & E = 362.0\ \text{N cm}^{-2}\ (525\ \text{psi}) \\ & \nu = 0.47 \end{array}\right\} \qquad (27)$$

where E is Young's modulus and ν is Poisson's ratio.

The bending moments $M(x)$ were calculated directly from the recorded fringe numbers. Then, noting that

$$V(x) = \frac{\mathrm{d}M}{\mathrm{d}x} \quad \text{and} \quad p(x) = \frac{\mathrm{d}V}{\mathrm{d}x} = \frac{\mathrm{d}^2M}{\mathrm{d}x^2}$$

the shearing forces in the beam $V(x)$ and contact pressure distribution $p(x)$ were determined by successive numerical differentiation of $M(x)$ with respect to x.

Durelli *et al.* presented the bending moment, beam shear and contact pressure distributions in the non-dimensional normalized form

$$\left.\begin{array}{l} M^*(x) = 4\lambda M(x)/P \\ V^*(x) = 2V(x)/P \\ p^*(x) = 2p(x)/P\lambda \end{array}\right\} \qquad (28)$$

where $\lambda = \sqrt[4]{k/4EI}$. They determined the k value needed by using the expression derived by Biot (1937), who matched the results for an infinite beam attached to a two-dimensional elastic continuum. According to their calculations, $k = 27.58$ N cm^{-2} (40 lb in^{-2}) and $\lambda = 0.1083$ cm^{-1} (0.275 in^{-1}).

This non-dimensionalization was found necessary by Durelli *et al.* since they compared the photoelastic results to a Winkler model. Because λ is a parameter that appears in the

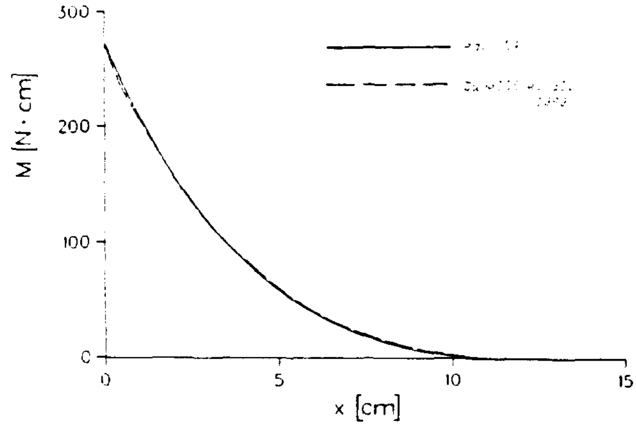

Fig. 3. Comparison of bending moments.

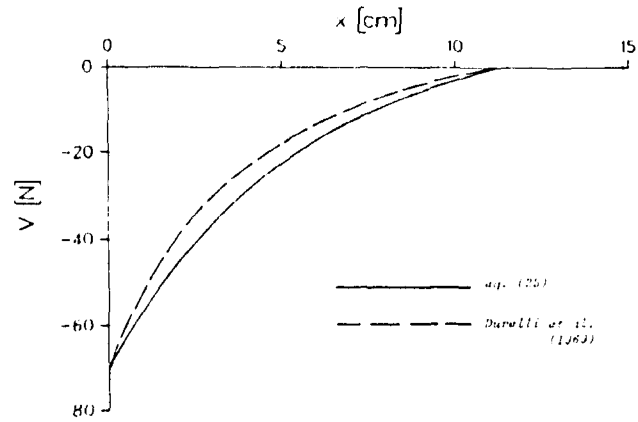

Fig. 4. Comparison of shearing forces in beam.

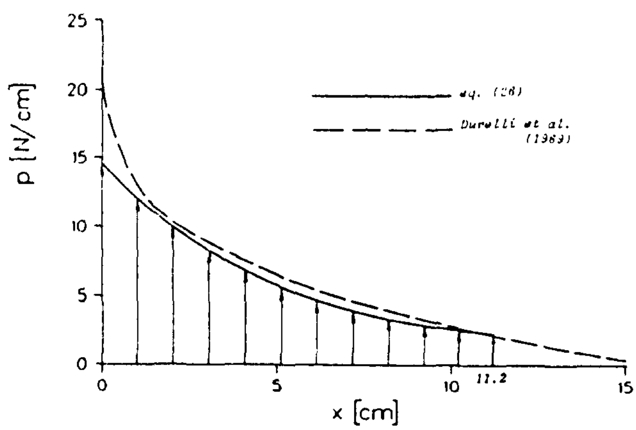

Fig. 5. Comparison of contact pressures.

Winkler foundation analysis and not in the one which utilizes a Pasternak base, as considered in the present paper, the non-dimensionalized graphs were converted back to their original form. The results are shown in Figs 3, 4 and 5, as dashed lines.

In a subsequent discussion Pandit (1970) pointed out that the contact pressure distribution, as determined by Durelli *et al.* (shown in Fig. 5), contains the inaccuracies inherent in the process of double differentiation of experimental results, and that this is most likely to be the cause of the sharp increase of the contact pressure distribution under the load. This possibility was acknowledged and accepted by Durelli *et al.* (1970).

When considering the contact pressure distribution in Fig. 5 it should be noted that vertical equilibrium must be satisfied. Thus,

$$\frac{P}{2} = \int_0^l p(x)\,\mathrm{d}x \tag{29}$$

must hold true. Whereas in the test $P/2 = 70$ N (31.46 lb), the integration of the area enclosed by the experimental contact pressure results in 92 N; this is 31% larger than $P/2$.

Another problem with the experimental pressure distribution is that the point of separation (beyond which the contact pressures are zero) corresponds to the end of the beam, 15.24 cm (6″). This contradicts their statement that when subjected to $P = 140$ N, the beam separated from the foundation at both ends. Furthermore, according to the test moment diagram shown in Fig. 3, the point of lift-off appears to be at $x = 11.2$ cm (4.5″) from the center of the beam. Therefore, the contact pressure distribution calculated from the bending moments recorded appears to be inaccurate near the load and should not exist past $x = 11.2$ cm.

Comparison of analytical and photoelastic results

In order to compare the analytic results based on the Pasternak foundation model presented previously with the photoelastic results, the Pasternak foundation parameters must be determined first. The methods for determining the foundation parameters were recently discussed by Kerr (1985). The criterion suggested for the determination of these parameters is that the analytical results agree as closely as possible with the actual situation, for the range of loads under consideration. It was utilized by Kneifati (1985).

Therefore, in the following the two Pasternak parameters are determined by collocating the test results with the corresponding analytical expressions. The number of collocation points to be used is equal to the number of unknown foundation parameters. Therefore, two data points are needed. For the problem under consideration the separation length l is obtained from the test.

Because of the small size of the test beam, it is reasonable to assume that the effect of the weight of the beam is negligible on the test response. This justifies the use of the presented analyses in the following comparative study. Since the foundation parameters are not known *a priori*, the analytical results for both cases $G < 2\sqrt{kEI}$ and $G > 2\sqrt{kEI}$ must be considered.

From the test curves in Figs 3, 4 and 5, the moment distribution is most accurate, since it was obtained directly from the test data. Therefore, it will be used for collocation purposes. From this curve two data points are chosen. They are:

$$\left.\begin{array}{lll} \text{at} & x_1 = 0\,\text{cm} & M_1 = 272\,\text{N cm (24.1 lb in)} \\ \text{at} & x_2 = 11.2\,\text{cm (4.4 in)} & M_2 = 0\,\text{lb in} \end{array}\right\}. \tag{30}$$

Substitution of these values into the moment expressions given in (15) and (24), noting that according to Fig. 3 the separation distance $l = x_2 = 11.2$ cm, results in two non-linear equations for the determination of k and G for each of the two cases $G \lessgtr 2\sqrt{kEI}$. Using the IMSL routine NEQNF and a Fortran program for the problem under consideration the foundation parameters are found to satisfy $G > 2\sqrt{kEI}$. They are:

$$k = 9.65\,\text{N cm}^{-2}\,(14\,\text{psi}), \quad G = 1864.5\,\text{N}\,(419\,\text{lb}). \tag{31}$$

It is of interest to note that this solution program when used on the two non-linear equations for $G > 2\sqrt{kEI}$ did not converge. It is also noteworthy that $l = x_2 = 11.2$ cm obtained from the test curve in Fig. 3 and the above two parameters k and G do satisfy eqn (23); the analytical condition for the determination of l.

The bending moments, shear forces and contact pressure distribution are then numerically determined using eqns (24)–(26). They are plotted as solid lines in Figs 3, 4 and 5.

According to Fig. 3 the photoelastic results and the analytical results agree very closely, throughout, for the beam bending moments, although only the two extreme points ($x = 0$ and $x = l$) were matched for the determination of k and G. The disagreement of the $V(x)$ curves and the $p(x)$ curves seem to be caused mainly by the inaccuracies introduced by the numerical differentiations of the M-test curves, as discussed previously. However, it should be noted that the slope discontinuity at the concentrated load P of the analytically-obtained $p(x)$ curve (Fig. 5) is another basic shortcoming† of the Pasternak model and is caused by the term Gw'' in eqn (2). It is reasonable to expect that in an actual situation the contact pressure distribution $p(x)$ will have a horizontal tangent at P: the point of symmetry.

It may be shown that the vertical equilibrium equation

$$P = 140\,\mathrm{N} = 2\int_0^l p(x)\,\mathrm{d}x \tag{32}$$

is satisfied for the analytical $p(x)$ expression, as anticipated.

CONCLUSIONS

Utilizing the variational approach for variable matching points derived by Kerr (1976), a formulation for the problem under consideration was presented that is mechanically reasonable and mathematically well posed. The analytical solution obtained was evaluated numerically and then compared with related photoelastic test results by Durelli *et al.* (1969).

It is noteworthy that although the two Pasternak foundation parameters were determined by collocating two extreme points on the test curve for bending moments, the agreement is very close throughout this curve. The agreement is only satisfactory for the beam-shearing force distribution $V(x)$ and the contact-pressure distribution p (except near the load P). It is pointed out that this disagreement seems to be caused mainly by the inaccuracies associated with the numerical differentiation of the M-test curve.

The discussion of the incorrect analyses by Chernigovskaya (1961) and Ting (1973), and the presented solution of the correct formulation show that caution has to be exercised when formulating problems of structures which rest on a Pasternak-type foundation.

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† In addition to the occurrence of concentrated reactions at the free beam ends that do not separate from base, as discussed by Kerr (1964).